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PERTURBATION ANALYSIS OF CONDENSATION CONTROLLED BY HEAT TRANSFER ON LARGE DROPLETS

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Abstract—The governing equations of condensation phenomena controlled by heat transfer on large droplets in a pure vapor are derived. Using the isentropic expansion as the first approximation, the equations are integrated by means of the perturbation method. This method is illustrated by presenting an example. The problem is an important one in the field of steam turbine engineering, although the condensation problem is of more general interest.

NOMENCLATURE

a,	radius of control sphere [m] or [μ];
c,	heat capacity of liquid [m^2/s^2degK];
c_p,	heat capacity of vapor at constant pressure [m^2/s^2deg$K$];
c_v,	heat capacity of vapor at constant volume [m^2/s^2deg$K$];
h,	latent heat [m^2/s^2];
k,	adiabatic index;
m,	mass of the sphere (vapor and liquid) [kg];
M,	Mach number;
p,	pressure [atm] or [kg/m s^2];
r,	spherical coordinate [m];
r_*,	radius of nucleus [m];
R,	gas constant [m^2/s^2deg$K$];
t,	time [s];
T_s,	saturation temperature at the beginning of expansion [deg$K$];
T_2,	saturation temperature on the droplet surface at a given pressure [deg$K$];
T_∞	temperature of vapor far from droplet [deg$K$];
u,	velocity [m/s];
y,	ratio of mass of liquid to mass of sphere;
δ,	radius of droplet [m] or [μ];
λ,	heat-conduction coefficient [kg m/s^3 deg$K$];
ρ,	density of vapor [kg/m^3];
ρ_2,	density of liquid [kg/m^3];
ω,	overcooling [deg$K$];
i_o, i_{o2},	constants in the relations for enthalpy [m^2/s^2];
i_2,	specific enthalpy of liquid [m^2/s^2];
i,	specific enthalpy of vapor [m^2/s^2];
U_2,	internal energy of liquid [m^2/s^2];
U,	internal energy of vapor [m^2/s^2];
p_2,	pressure inside the droplet [atm] or [kg/m s^2].

INTRODUCTION

THE PROBLEM of condensation on droplets appears in a variety of forms as the object of scientific and engineering research. This problem is, among others, an important one in steam turbine engineering.

The behavior of liquid phase is an essential factor in determining the work of the last stages of steam turbines. The desire to increase the efficiency of these stages yields a growing

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interest in phenomena associated with wet steam expansion. Among the many aspects of this problem the process of condensation is of great importance. The condensation process itself is interesting for general reasons, especially in the investigation of two-phase flow.

The phenomenon of "condensation shock" has mainly been investigated [1, 2, 3] until now in connection with steam expansion. The region of "condensation shock" is that in which the non-equilibrium state breaks down very rapidly and the liquid phase appears, resulting in a characteristic change of parameters which resembles a "washed out" shock. The droplets formed in this region due to nucleation are very small in comparison with the mean free path.

In addition to these small droplets, large drops are formed during the expansion in cascades of blades, either as the result of steam condensation on small droplets, or due to disintegration of liquid films covering the surfaces of turbine blades.

In this paper a theoretical analysis of condensation on large droplets has been carried out by means of the perturbation method. As far as it is known to the author, recent methods of calculation of the expansion in blade rows are based on equilibrium thermodynamics. The method described below allows us to predict the deviation from equilibrium during expansion for a given dispersion of liquid phase.

While preparing the present paper the author faced the dilemma, whether to construct a more accurate model of phenomena of which an analytic solution would not be feasible, or to make further simplifying assumptions and use a model capable of calculation. By choosing the second method, the author believes that the results, from an engineering point of view, are more useful.

THE MAIN ASSUMPTIONS

Droplets large in comparison with mean free path are considered, the latent heat is assumed to be released from droplets to the surrounding atmosphere of pure vapor due to heat transfer of strictly macroscopic character. There is a coupling between heat transfer and mass transport which controls the growth of the droplets.

The upper limit of mass transport to the droplet can be estimated from equilibrium condensation using, for example, the entropy–enthalpy diagrams. The velocity of condensed mass near the surface during equilibrium expansion at a practically reasonable rate is of the order of 10^{-2}–10^{-3} m/s for droplet radius of about 1 μ. For such a small velocity we neglect the propagation of momentum and convection component of heat transfer. This leads to the spatially uniform pressure at any instant

$$\frac{\partial p}{\partial r} = 0,$$

and heat transfer is primarily by conduction. It is conceivable that these assumptions may not be justified when the slip velocity of a droplet with respect to vapor is large. However, if the velocity equilibration time of droplet is very small in comparison to the time of expansion we can neglect the influence of slip velocity on the condensation process. For example, for the droplet radius 1 μ the velocity equilibration time in pure vapor is of the order of 10^{-5} s.

Further simplifying assumptions concern the temperature of the droplets. We assume that heat conduction inside the droplet is infinite. This assumption is reasonable when the thermal equilibration time of a droplet is small in comparison with expansion time. The thermal equilibration time for droplet radius 1 μ is of the order of 10^{-5} s. Assuming the local equilibrium near the droplet surface, the temperature of the droplet will follow the saturation temperature for a given pressure. Using the assumption about quasi-stationary process of heat conduction from a droplet to vapor, the equation of temperature reduces itself to

$$\frac{\partial}{\partial r}\, r^2 \frac{\partial T}{\partial r} = 0$$

with boundary conditions

$$r = \delta, \qquad T = T_2 = f(p),$$
$$r = \infty, \qquad T = T_\infty.$$

This is the assumption commonly used in the investigation of two phase flow in nozzles [4, 5].

If the equation of unsteady heat transfer is considered, which reads

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$$

and the following values are taken for the estimation

$$\alpha = 0{\cdot}3\,.\,10^{-4}\,\mathrm{m^2/s}, \qquad \Delta r = 10^{-6}\,\mathrm{m},$$
$$\Delta T = 100\mathrm{degK}, \qquad \Delta t = 5\,.\,10^{-4}\mathrm{s},$$

we get the order of 10^9 for the right-hand side and the order of 10^5 for the left-hand side. Thus the assumption of the quasistationary character of heat conduction seems to be reasonable under certain conditions.

Consequently we get simple, well-known, solutions for the distribution of temperature and pressure around the droplet

$$T = (T_2 - T_\infty)\frac{\delta}{r} + T_\infty \tag{1}$$

$$p = p(t). \tag{2}$$

In order to find the deviation from equilibrium due to the condensation on large droplets, we neglect the nucleation phenomenon even if it occurs as the result of overcooling in the region between droplets. We shall turn back to the estimation of this effect in what follows.

GOVERNING EQUATIONS

We will consider the expansion of a vapor with radius $a(t)$ which contains a droplet of radius $\delta(t)$ at the center (Fig. 1). The initial values $a(0)$ and $\delta(0)$ depends on initial dispersion of liquid phase and initial moisture of the vapor.

We will assume that there is no change in the spherical shape of the atmosphere surrounding the droplet during the expansion. By this we avoid having complications with the deformation of the flow. Let us write the conservation of mass for liquid and vapor in the forms

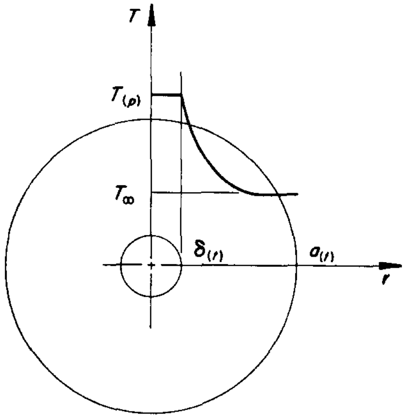

FIG. 1. Droplet and control sphere.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4}{3}\pi\delta^3\rho_2 + 4\pi\int_\delta^a \rho r^2\,\mathrm{d}r\right] = 0 \tag{3}$$

and the first thermodynamic law with the assumption that the sphere is expanding adiabatically (no heat addition from outside the sphere)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4}{3}\pi\delta^3\rho_2 U_2 + 4\pi\int \rho U r^2\,\mathrm{d}r\right] + p_2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi\delta^3\right) + p\frac{\mathrm{d}}{\mathrm{d}t}\left[4\pi\int_\delta^a r^2\,\mathrm{d}r\right] = 0. \tag{4}$$

Assuming that the vapor is a perfect gas

$$p = \rho RT. \tag{5}$$

As was mentioned above, the change of internal energy of the droplet results from heat conduction and mass transfer, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi\delta^3\rho_2 i_2\right) = i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi\delta^3\rho_2\right) - \dot{q}, \tag{6}$$

where

$$\dot{q} = -\lambda 4\pi\delta^2 \left(\frac{\partial T}{\partial r}\right)_{r=\delta} \quad (7)$$

$$i_2 = cT_2 + i_{o2}$$

$$i = c_p T + i_o$$

which gives after substituting (1) into (7) and then into (6)

$$\lambda(T_2 - T_\infty) = \rho_2 h\delta \frac{\mathrm{d}\delta}{\mathrm{d}t} - \frac{1}{3}\rho_2 c \frac{\mathrm{d}T}{\mathrm{d}t}\delta^2, \quad (8)$$

where

$$h = i - i_2$$

is the latent heat.

Integrating the Clapeyron–Clausius relation,

$$\frac{\mathrm{d}p}{p} = \frac{h}{R}\frac{\mathrm{d}T_2}{T_2^2} \quad (9)$$

with the assumption that h = constant and that at any pressure $p = p_s$, we know the saturation temperature $T_2 = T_s$, we obtain

$$T_2 = \frac{T_s}{1 - \frac{RT_s}{h}\ln\frac{p}{p_s}}. \quad (10)$$

In order to simplify the relations, let us choose an expression for pressure change as

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \theta(t) \quad (11)$$

so that, at $t = 0$, $p = p_s$, we get

$$p = p_s \exp\left[\int_0^t \theta(t)\,\mathrm{d}t\right]. \quad (12)$$

At moderate temperatures, the magnitude of expression RT_s/h for water vapor in relation (10) is of the order 0·1 or less. Therefore, for not very large expansion ratios, we can change relation (10) into the form

$$T_2 = T_s\left(1 + \frac{RT_s}{h}\int_0^t \theta(t)\,\mathrm{d}t\right). \quad (13)$$

Substituting (13) into (8) we obtain equations for $\bar{T}$ and $\bar{\delta}$, which are defined as

$$T_\infty = \bar{T}\,T_s, \qquad \delta = \bar{\delta}\,\delta_0,$$

in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\delta}^2 - \bar{n}\theta(t)\,\bar{\delta}^2 = \frac{1}{\tau}\left(1 + \bar{m}\int_0^t \theta(t)\,\mathrm{d}t - \bar{T}\right), \quad (14)$$

where we introduce the dimensionless numbers

$$\bar{n} = \frac{2}{3}\frac{cRT_s^2}{h^2}, \qquad \bar{m} = \frac{RT_s}{h}$$

and the characteristic time

$$\tau = \frac{\rho_2 h\delta_0^3}{2\lambda(T_s/\delta_0)\,\delta_0^2}.$$

Then τ is the time required to transport heat $\rho_2 h\delta_0^3$ through the surface δ_0^2 due to the temperature gradient T_s/δ_0 and heat conductivity λ.

In order to derive an equation for $\bar{T}$ we can rearrange the relation (4), applying (3) to the form

$$\frac{1}{\tilde{\rho}}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\mathrm{d}\tilde{i}}{\mathrm{d}t} + \frac{\tilde{i} - i_2}{4\pi\int_\delta^a \rho r^2\,\mathrm{d}r}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi\delta^3\rho_2\right) = \frac{\frac{4}{3}\pi\delta^3\rho_2}{4\pi\int_\delta^a \rho r^2\,\mathrm{d}r}\left(\frac{1}{\rho_2}\frac{\mathrm{d}p_2}{\mathrm{d}t} - \frac{\mathrm{d}i_2}{\mathrm{d}t}\right). \quad (15)$$

We introduce in the above equation two average quantities

$$\tilde{\rho} = \frac{4\pi\int_\delta^a \rho r^2\,\mathrm{d}r}{4\pi\int_\delta^a r^2\,\mathrm{d}r}, \quad (16)$$

and

$$\tilde{i} = \frac{4\pi\int_\delta^a i\rho r^2\,\mathrm{d}r}{4\pi\int_\delta^a \rho r^2\,\mathrm{d}r} = \frac{c_p p}{R\tilde{\rho}} + i_0. \quad (17)$$

Using the equation of state (5) and the relation for the temperature distribution (1) we obtain

$$\tilde{\rho} = \frac{p}{RT_\infty}\left[1 - \frac{3}{2}\left(\frac{T_2}{T_\infty} - 1\right)\frac{\delta}{a} + 0\left(\frac{\delta^2}{a^2}\right)\right]. \tag{16a}$$

Neglecting the term

$$\frac{3}{2}\left(\frac{T_2}{T_\infty} - 1\right)\cdot\frac{\delta}{a}$$

we have

$$\tilde{\rho} = \frac{p}{RT_\infty}.$$

In most applications the mass of liquid is much smaller than the mass of vapor, so

$$\frac{4}{3}\pi\delta^3\rho_2 \ll 4\pi\int_\delta^a \rho r^2\,\mathrm{d}r.$$

If we assume further that

$$c_p = \text{const}$$

and

$$\tilde{i} - i_2 \cong i - i_2 = h,$$

we can reduce equation (15) to the form

$$-\frac{k-1}{k}\bar{T}\,\theta(t) + \frac{\mathrm{d}}{\mathrm{d}t}\bar{T} - \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\bar{\delta}^3 = 0, \tag{18}$$

where the dimensionless number

$$\varepsilon = \frac{4\pi\rho_2 h\delta_0^3}{3mc_pT_s}.$$

Finally, we obtain two equations (14) and (18) which describe the behavior of $\bar{\delta}$ and $\bar{T}$ during the condensation process on a droplet within the limitations of above assumptions.

Starting from the equilibrium state, the initial conditions at $t = 0$ are

$$\bar{T} = 1 \quad \text{and} \quad \bar{\delta} = 1.$$

PERTURBATION METHOD OF SOLVING THE EQUATIONS

The perturbation method of solving the equations involved in two-phase problems has been used successfully in [4] and [5]. Depending on the physical aspect of the phenomena, different groups of parametres involved in the problems have been used. The main point of such a method lies in finding reasonable "zero order solution" of the problem and a perturbation parameter.

Looking at the equations derived above,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\delta}^2 - \bar{n}\theta(t)\,\bar{\delta}^2 = \frac{1}{\tau}\left(1 + \bar{m}\int_0^t \theta(t)\,\mathrm{d}t - \bar{T}\right) \tag{19a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{T} - \frac{k-1}{k}\theta(t)\,\bar{T} = \bar{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\bar{\delta}^3, \tag{19b}$$

we can find two extremal solutions with regard to the behavior of two characteristic parameters τ and $\bar{\varepsilon}$.

When the characteristic time $\tau = 0$ (i.e. $\lambda = \infty$), we have, from (19a),

$$\bar{T} = 1 + \bar{m}\int_0^t \theta(t)\,\mathrm{d}t. \tag{20}$$

This is the case of equilibrium condensation.

From equation (19b) we obtain

$$\bar{\delta} = \left\{1 + \frac{1}{\varepsilon}\int_0^t \bar{m}\theta(t) - \frac{k-1}{k}\theta(t) \times \left(1 + \bar{m}\int_0^t \theta(t)\,\mathrm{d}t\right)\mathrm{d}t\right\}^{\frac{1}{3}}. \tag{21}$$

When $\bar{\varepsilon} = 0$ (this means that the influence of the existing droplet is negligible in expansion of the sphere), the temperature of the vapor changes as in isentropic expansion; we have from (19b)

$$\bar{T} = \exp\left(\frac{k-1}{k}\int_0^t \theta(t)\,\mathrm{d}t\right). \tag{22}$$

we get from (20)

$$\bar{T} = 1 - 1{\cdot}492 \times 10^{2}\, t \tag{36}$$

and from (21),

$$\bar{\delta} = (1 + 3{\cdot}54 \times 10^{3}\, t - 3{\cdot}9 \times 10^{5}\, t^{2})^{\frac{1}{3}}. \tag{37}$$

The agreement with values obtained by means of diagrams is quite satisfactory, as is shown in

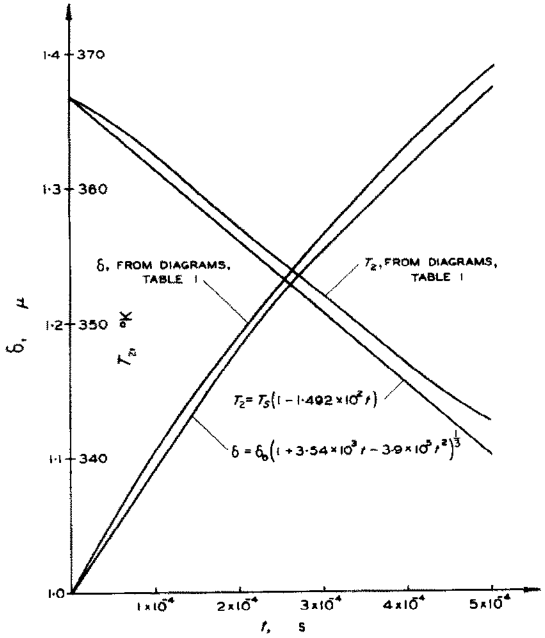

FIG. 3. Comparison of analytical results with results obtained from s–i diagram.

Fig. 3. For zero approximation from (22), we have

$$\bar{T}_0 = \exp(-4{\cdot}62 \times 10^{2}\, t), \tag{38}$$

and after expansion in powers of t, from (23) we get

$$\bar{\delta}_0 = (1 - 67{\cdot}5\, t + 1{\cdot}18 \times 10^{6}\, t^{2})^{\frac{1}{3}}. \tag{39}$$

The results of the calculation are shown in Fig. 4. Also shown is the influence of the terms of the

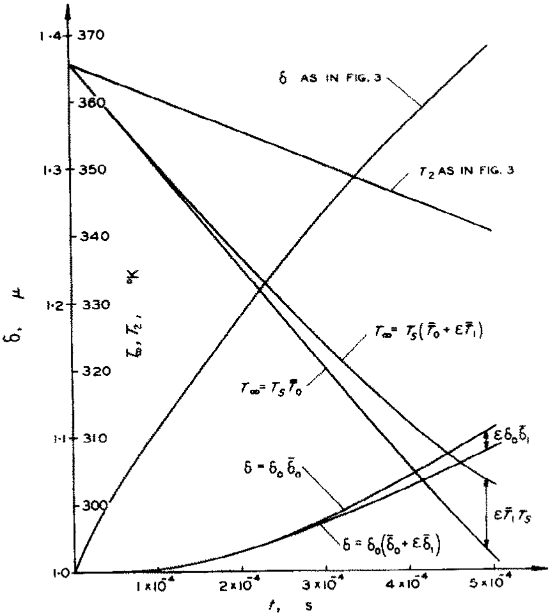

FIG. 4. Temperature and droplet radius for zero- and first-order solutions.

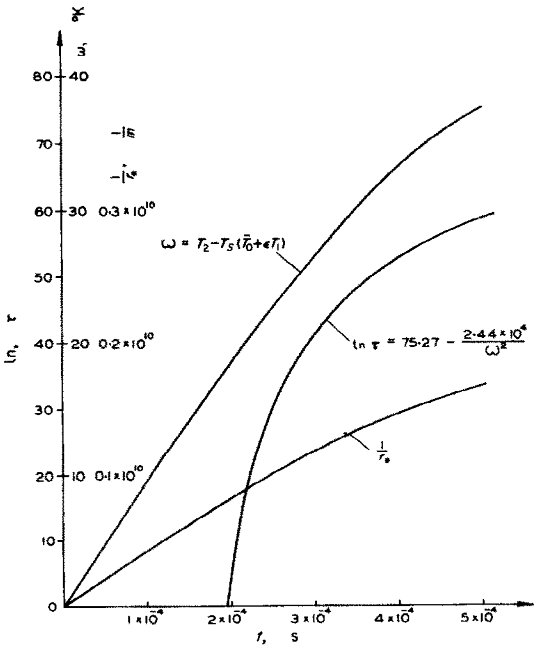

FIG. 5. Behavior of the condensation factors during ex-

been found numerically. The corrections connected with the second approximation are negligible in this case.

The deviation from the equilibrium state given by (31) is shown in Fig. 5. On this same figure are shown nuclei forming rate (32) and radius of nucleus (33). The growth of the relative moisture in the equilibrium expansion case is shown in Fig. 6, also moisture caused by condensation controlled by heat transfer, y_λ, and moisture y^* liberated by the nucleation effect.

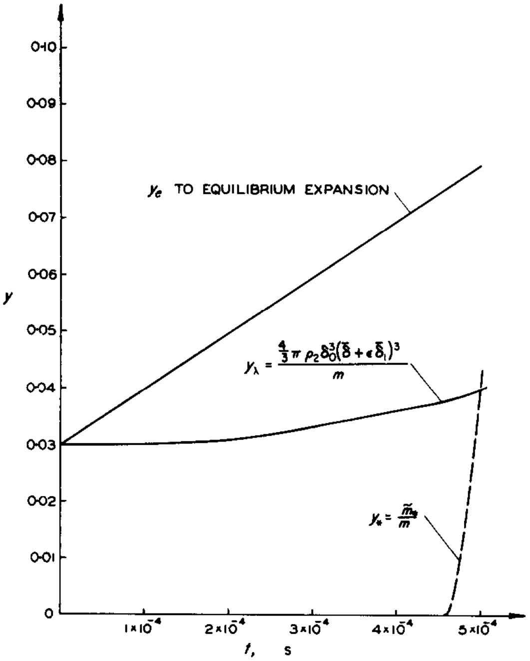

FIG. 6. Growth of moisture during expansion.

At the end of the expansion, the relative moisture y^* is quite appreciable in comparison with the moisture y_λ.

It is also worth mentioning that in this case the correction in (16),

$$\frac{3}{2}\left(\frac{T_2}{T_\infty} - 1\right)\frac{\delta}{a},$$

is very small. At the end of the expansion, this correction is approximately 0·0038.

CONCLUSIONS

With assumptions that seem reasonable from the practical point of view, it is possible to solve the system of equations describing the growth of large droplets and the change of temperature of the vapor.

A solution can be found either for the case of equilibrium condensation or for the more realistic case, i.e. condensation controlled by heat conductivity. In this case, the isentropic expansion of the vapor is the first approximation in the perturbation method. As was shown in the example, the change of temperature of the vapor is closer to the isentropic expansion than to the equilibrium expansion. It causes overcooling and nucleation as well. Therefore, in some cases we can expect sufficient nucleation zone (so-called condensation shock) regardless of whether the vapor contains moisture or not at the beginning of expansion.

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Résumé—Les équations de base des phénomènes de condensation par le transport de chaleur sur de grosses gouttelettes dans une vapeur pure sont obtenues. En employant la détente isentropique en première approximation, les équations sont intégrées par la méthode des perturbations. Le problème est important dans le domaine de la technique des turbines à vapeur bien que le problème de la condensation soit d'un intérêt plus général.

Zusammenfassung—Es wurden die kennzeichnenden Gleichungen abgeleitet für die Kondensation an einem grossen Tropfen in reinem Dampf. Unter Benutzung der isentropen Expansion als erste Näherung wurden die Gleichungen mit Hilfe der Strömungsmethode integriert. Diese Methode wird an einem Beispiel dargestellt. Das Problem ist wichtig in der Dampfturbinentechnik, daneben ist das Kondensationsproblem von allgemeinem Interesse.

Аннотация—Выведены основные уравнения для конденсации, контролируемой теплообменом на больших каплях в чистом паре. Уравнения интегрируются методом возмущений с использованием изэнтпорного расширения к вачестве первого приближения. Задача играет важную роль в паро-турбинной технике, хотя проблема конденсации представляет более общий интерес.